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On different approaches to obtaining partial effects in binary response models with endogenous regressors

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HIGHLIGHTS

- We study partial effects in binary response models.
- Our interest is in models with endogenous explanatory variables.
- The average structural function is superior to the average index function.
- Our conclusions hold for linear models and probit models.

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ABSTRACT

We compare three different approaches to obtaining partial effects in binary response models. Among the three approaches, we maintain that the average structural function (ASF) due to Blundell and Powell (2003, 2004) defines the marginal effect of primary interest, for it is based on the unconditional marginal distribution of the structural error. Analytical examples are provided to show that the average index function (AIF) proposed recently by Lewbel, Dong, and Yang (2012), suffers from essentially the same shortcomings as the propensity score as a basis for defining average partial effects.

1. Introduction

Defining the magnitude of partial effects in nonlinear models is now widely recognized as being important in empirical studies. A leading case is when the response variable is binary and one wants to allow a nonlinear response function. When the explanatory variables are exogenous, a natural approach to defining partial effects is in terms of the response probability. Specifically, if D is the binary response and $\mathbf{X}$ is a vector of exogenous explanatory variables, the partial effects are usually defined in terms of $p(\mathbf{x}) \equiv P(D = 1|\mathbf{X} = \mathbf{x})$. The function $p(\cdot)$ is sometimes called the propensity score (PS), in an effort to link the econometrics literature and the potential outcomes literature (for example Rosenbaum and Rubin, 1983). When one would like a single number to describe the effect of a variable, say X_j, on $p(\mathbf{x})$, it is common to average the partial derivative or discrete change across the distribution of $\mathbf{X}$. This yields what is often called the average partial effect (APE) (or average marginal effect). Wooldridge (2010, Section 15.6) contains a general discussion.

When a binary response model includes one or more endogenous explanatory variables (EEVs), one must use care in defining interesting partial effects. At a minimum, the definition should reduce to what is widely accepted as the quantity of interest in a linear model. Partial effects based on the propensity score are rarely of interest, as that corresponds to computing $E(Y|\mathbf{X})$ in models where $\mathbf{X}$ contains endogenous variables. If $E(Y|\mathbf{X})$ were the object of interest then decades of published research on accounting for EEVs in econometric models would be irrelevant.

Partial effects based on the average structural function (ASF), as defined by Blundell and Powell (2003, 2004), have the desirable feature that they correspond to the parameters of interest in linear models with EEVs. In addition, in common nonlinear models they are intuitively appealing and can be obtained via counterfactual reasoning.

To define the ASF, let Y be the response variable, $\mathbf{X}$ a set of explanatory variables (possibly containing endogenous as well as exogenous explanatory variables), and $\mathbf{U}$ a vector of unobservables.

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Assume that $Y = g(\mathbf{X}, \mathbf{U})$ for some function $g(\cdot, \cdot)$. The ASF is defined as

$$ASF(\mathbf{x}) \equiv E_{\mathbf{U}}\left[g(\mathbf{x}, \mathbf{U})\right], \tag{1}$$

where $\mathbf{x}$ is a nonrandom placeholder. In other words, the ASF averages out the unobservables, $\mathbf{U}$, across the population and is a function of potential values of the observed explanatory variables, $\mathbf{x}$. It takes no stand on whether $\mathbf{U}$ and $\mathbf{X}$ are dependent. One can then define partial effects based on $ASF(\mathbf{x})$ using partial derivatives or discrete differences. See Wooldridge (2010, Sections 2.2.5 and 15.7.4) for further discussion, including how standard methods of computing APEs can be obtained by first defining the ASF. If we have a single binary covariate X, and its values represent control and treatment states, then $ASF(1) - ASF(0) = E_{\mathbf{U}}\left[g(1, \mathbf{U})\right] - E_{\mathbf{U}}\left[g(0, \mathbf{U})\right]$ is the average treatment effect, as in Rosenbaum and Rubin (1983).

Recently, Lewbel et al. (2012) (LDY) have proposed a different approach to estimating partial effects for a class of semiparametric index binary response models. More precisely, LDY define a new function called the average index function (AIF). LDY argue that the AIF is a "middle ground" between the propensity score and the ASF. In the LDY setup the model has an index structure, so that the binary response, D, depends on $\mathbf{X}$ through a linear index $\mathbf{X}\beta$, where β is a column vector of parameters. Because $\mathbf{X}\beta$ is a scalar, estimation of the AIF is simplified compared with fully nonparametric estimation of the average structural function. As discussed by LDY, the AIF can be identified generally under an index structure in the presence of a "special regressor". By contrast, identification of the ASF requires additional assumptions when some endogenous elements of $\mathbf{X}$ are discrete (although it does not assume a special regressor).

As noted by LDY, the definition of the AIF does not rely on the special regressor setup; the presence of the special regressor simply allows one to identify the index coefficients β. Here we are interested in interpreting the AIF. Therefore, in most of what follows we do not separately show a special regressor. Then we can write an index model as

$$D = 1\left[\mathbf{X}\beta + U \geq 0\right], \tag{2}$$

where $1[\cdot]$ denotes the indicator function and $(\mathbf{X}, U)$ is a random vector.

Given the index structure in (2), three functions that can be used to define partial effects are the propensity score, AIF, and ASF:

$$PS(\mathbf{x}) = F_{-U|\mathbf{X}}(\mathbf{x}\beta|\mathbf{x}), \tag{3a}$$

$$AIF(\mathbf{x}) = F_{-U|\mathbf{X}\beta}(\mathbf{x}\beta|\mathbf{x}\beta), \tag{3b}$$

$$ASF(\mathbf{x}) = F_{-U}(\mathbf{x}\beta), \tag{3c}$$

where $F(\cdot)$ denotes a cumulative distribution function. Note that $ASF(\mathbf{x})$ depends only on the unconditional CDF of $-U$. In thinking about $AIF(\mathbf{x})$, note that it can be obtained by first defining the random variable $W = \mathbf{X}\beta$ and then obtaining the CDF of $-U$ given W. Then, one replaces the placeholder, w, with $\mathbf{x}\beta$. The propensity score, $PS(\mathbf{x})$, depends on the CDF of $-U$ given $\mathbf{X}$. Generally, PS, AIF, and ASF all differ, although it follows immediately that they are all the same when U and $\mathbf{X}$ are independent. Blundell and Powell (2003) suggest estimating partial effects from $ASF(\mathbf{x})$ whereas LDY suggest using $AIF(\mathbf{x})$.

It is widely agreed that $PS(\mathbf{x})$ is usually not of interest when $\mathbf{X}$ has elements correlated with U: the partial effects obtained from $PS(\mathbf{x})$ generally have nothing to do with the causal effects. By contrast, as argued by Blundell and Powell (2003, 2004), $ASF(\mathbf{x})$ is of considerable interest whether or not $\mathbf{X}$ includes EEVs; see also Wooldridge (2010, Section 2.2).

LDY argue that AIF is a useful "middle ground" between the propensity score and ASF. The purpose of this note is to show that in standard cases the AIF suffers from essentially the same shortcomings as the propensity score because it is affected by correlation between the unobservables and the observed EEVs. In fact, in some simple cases with endogenous variables the AIF and propensity score are the same.

In Section 2 we use a linear model to illustrate the relationships among the three partial effects. Section 3 covers binary response models and shows that, under standard assumptions, the AIF and propensity score are identical. Section 4 contains a brief conclusion.

2. A linear model

We start with a linear model to show how the AIF does not identify quantities of interest in the most common setting. Consider the model

$$Y = X\beta + U, \tag{4}$$

where Y, X, and U are all scalars for simplicity. Nothing substantive changes if we include an intercept or allow multiple explanatory variables. Assume that (X, U) has a zero mean bivariate normal distribution:

$$\begin{pmatrix} X \\ U \end{pmatrix} \sim Normal\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_U \\ \rho\sigma_X\sigma_U & \sigma_U^2 \end{pmatrix}\right], \tag{5}$$

where $\rho = Corr(X, U)$ is the correlation coefficient. The average structure function in this model is simply

$$ASF(x) = x\beta, \tag{6}$$

because $E(U) = 0$. It is important to understand that the ASF has nothing to do with the conditional distribution of U given X. Whether (4) represents a single equation in a simultaneous system or U contains an omitted variable correlated with X, all economists would agree that β is the quantity of interest.

For this simple linear model, the propensity score is analogous to $E(Y|X = x)$. The analog of the AIF is to obtain

$$E(Y|X\beta) = X\beta + E(U|X\beta), \tag{7}$$

and then to plug in x. Assuming that $\beta \neq 0$ to rule out degeneracies, by joint normality,

$$E(U|X\beta) = \tau\,(X\beta), \tag{8}$$

where the scalar τ is defined as

$$\tau = \frac{Cov(X\beta, U)}{Var(X\beta)} = \frac{\beta Cov(X, U)}{\beta^2\sigma_X^2} = \frac{\rho\sigma_U}{\beta\sigma_X}. \tag{9}$$

Plugging this expression for τ into (8) gives

$$E(U|X\beta) = \frac{\rho\sigma_U}{\sigma_X}X, \tag{10}$$

and so

$$AIF(x) = x\left(\beta + \frac{\rho\sigma_U}{\sigma_X}\right) = E(Y|X = x). \tag{11}$$

In this simple linear case, the AIF is identical to the conditional expectation, $E(Y|X = x)$. As in almost any case with endogenous explanatory variables, the conditional mean is not of interest. We have shown explicitly in (11) that the partial effect defined in terms of the AIF does not deliver the effect of interest, β.

If we add exogenous variables to the model then the link between the conditional expectation and AIF is broken, but the latter is still of doubtful interest. Let V be a normally distributed random variable independent of (X, U). Now assume

$$Y = X\beta + V\gamma + U, \tag{12}$$

which means the average structural function is $ASF(x, v) = x\beta + v\gamma$, and so the partial effect of x on the ASF is β. Under the same assumptions as before, it is easily shown that

$$E(Y|X\beta + V\gamma) = X\left(\beta + \frac{\rho\sigma_U\sigma_X}{\sigma_X^2 + \gamma^2\sigma_V^2/\beta^2}\right) + V\left(\gamma + \frac{\rho\sigma_U\sigma_X}{\beta\sigma_X^2/\gamma + \gamma\sigma_V^2/\beta}\right), \tag{13a}$$

$$E(Y|X, V) = X\left(\beta + \frac{\rho\sigma_U\sigma_X}{\sigma_X^2}\right) + V\gamma. \tag{13b}$$

When $\gamma \neq 0$ (as in the special regressor setup, where $\gamma = 1$), the inconsistency in the APE of X defined by $AIF(x)$ is less than that based on $E(Y|X, V)$. Perhaps this is the sense in which AIF constitutes a "middle ground" between the conditional expectation and the ASF. Nevertheless, the AIF still does not identify the parameter of interest, β. And, unlike the conditional expectation, the AIF gets the coefficient on exogenous variable V wrong, too.

3. A binary response model

Now we turn to a binary response model, the context in which LDY defined the AIF. As with the linear case, the presence of a special regressor has nothing to do with the definition of the AIF. Therefore, we do not explicitly include a special regressor.

Specify the binary response D as

$$D = 1[X\beta + U \geq 0]. \tag{14}$$

If U and X are independent, it is easy to see that the PS, AIF, and ASF are all the same, and equal to

$$ASF(x) = P(-U \leq x\beta) = F_{-U}(x\beta). \tag{15}$$

Assuming that $F_{-U}(\cdot)$ is differentiable with density $f_{-U}(\cdot)$, for a continuous X, the partial effect is

$$\frac{\partial ASF(x)}{\partial x} = \beta f_{-U}(x\beta), \tag{16}$$

which has the same sign as β. Importantly, Eqs. (15) and (16) remain valid for any dependence between U and X. Therefore, it is interesting to see what happens to AIF (and PS) when we allow various forms of dependence between U and X. The following considers two cases that U and X are not independent.

Without specifying the unconditional density of U, suppose that

$$U|X \sim Normal[0, \exp(2X\beta)], \tag{17}$$

so that U has a zero mean conditional on X but is heteroskedastic, with its variance depending on $X\beta$. Then, with $\Phi(\cdot)$ the standard normal CDF,

$$P(-U \leq X\beta|X\beta) = P\left[\frac{-U}{\exp(X\beta)} \leq \frac{X\beta}{\exp(X\beta)}\middle| X\beta\right] = \Phi[\exp(-X\beta)X\beta], \tag{18}$$

because $-U/\exp(X\beta)$ is independent of X with a standard normal distribution. With $\beta \neq 0$ we get the same expression if we condition on X rather than $X\beta$. Therefore,

$$AIF(x) = PS(x) = \Phi[\exp(-x\beta)x\beta]. \tag{19}$$

Letting $\phi(\cdot)$ be the standard normal PDF, the partial effect calculated from the AIF is

$$\frac{\partial AIF(x)}{\partial x} = \beta\exp(-x\beta)(1 - x\beta)\phi[\exp(-x\beta)x\beta], \tag{20}$$

which has the same sign as β only when $x\beta < 1$. Compared with the ASF, the partial effect based on the AIF is more complicated, and it does not always have the same sign as β. The partial effect based on the ASF is simply a scaled version of β, so its properties are more appealing and more in the spirit of what we expect with a linear index model.

If X is correlated with U then, with the standard binary response model normalization $\sigma_U^2 = 1$, the ASF in (15) is still valid. The APE based on the ASF is

$$APE_X^{(ASF)} = \beta E_X[\phi(X\beta)], \tag{21}$$

which has the same sign as β because $\phi(z) > 0$ for all $z \in \mathbb{R}$.

For the AIF, we follow the derivation for the linear model. By joint normality we can write the structural error U as

$$U = \tau(X\beta) + R, \tag{22a}$$

$$R|X \sim Normal(0, \sigma_R^2), \tag{22b}$$

$$\tau = \frac{\rho\sigma_U}{\beta\sigma_X}, \tag{22c}$$

$$\sigma_R^2 = 1 - \tau^2\beta^2\sigma_X^2 = 1 - \rho^2. \tag{22d}$$

Note that R is independent of X, and thus of $X\beta$. Therefore,

$$D = 1[(1+\tau)X\beta + R \geq 0], \tag{23}$$

and

$$E(D|X\beta) = P(-R \leq (1+\tau)X\beta|X\beta) = P\left[\frac{-R}{\sigma_R} \leq \frac{(1+\tau)X\beta}{\sigma_R}\middle| X\beta\right] = \Phi\left[\frac{(1+\tau)X\beta}{\sigma_R}\right]. \tag{24}$$

It follows that the average partial effect based on the AIF is

$$APE_X^{(AIF)} = \frac{(1+\tau)\beta}{\sigma_R}E_X\left\{\phi\left[\frac{(1+\tau)X\beta}{\sigma_R}\right]\right\}, \tag{25}$$

which has the same sign as β only if $1 + \tau > 0$. In fact, it is easy to choose values of the population parameters so that $\tau = -1$, in which case $APE_X^{(AIF)} = 0$ for any sign or magnitude of β. Alternatively, $APE_X^{(AIF)}$ could be much larger than $APE_X^{(ASF)}$. Either way, it is hard to see how $APE_X^{(AIF)}$ is of any use because it has little to do with structural or causal effects.

4. Conclusion

We have compared three definitions of average partial effects in binary response models that contain endogenous explanatory variables. As is well known, the propensity score is not useful for summarizing partial effects when some explanatory variables are correlated with the unobservables. APEs based on the average structural function of Blundell and Powell (2003) have proven to be useful in both parametric and nonparametric approaches. Recently, Lewbel et al. (2012) introduced the average index function for a class of semiparametric binary response models, and estimation of the AIF has been implemented in Stata (Baum, 2012). Unfortunately, we have shown that AIF has essentially the same shortcomings as the propensity score.

While the ASF is a generally useful concept, it is not always nonparametrically or semiparametrically identified in models with discrete EEVs. Ongoing research, for example Chesher (2010), succeeds in bounding average partial effects in certain models with discrete EEVs. Chesher focuses on the structural function, and it is difficult to see how the AIF will have a role in searching for useful bounds.

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